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DISCONTINUOUS SOLUTIONS OF THE "SHALLOW WATER" EQUATIONS FOR FLOW OVER A BOTTOM STEP

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Flows over a depth discontinuity (bottom step) are studied within the framework of a single-layer "shallow water" model. Emphasis is given to substantiation of the relations for the stationary discontinuity thus formed. Admissible stable flows over this discontinuity are distinguished. As an example, the paper gives a solution of the problem of the water flow resulting from dam failure above a bottom step over which water is flowing.

1. Formulation of the Problem. For the case of a rectangular channel of constant width and variable depth, the differential single-layer "shallow water" equations (Saint Venant equation) [1–4] ignoring friction are written as

$$h_t + q_x = 0; \tag{1.1}$$

$$q_t + (qv + h^2/2)_x = -hb_x, \tag{1.2}$$

where $h(x,t)$ is the flow depth, $q(x,t)$ is the flow rate, $v = q/h$ is the fluid velocity, and $b(x)$ is the bottom depth. The free-fall acceleration is $g = 1$. In classical "shallow water" theory, the bottom depth $b(x)$ is assumed to be a smooth function with a finite derivative. In this case, Eqs. (1.1) and (1.2) are the basic conservation laws [3], which lead to the Hugoniot shock condition:

$$D[h] = [q]; \tag{1.3}$$

$$D[q] = [qv + h^2/2], \tag{1.4}$$

where D is the shock propagation velocity and $[f] = f_1 - f_0$ is the jump of the function f at its front.

We assume that the piecewise constant function $b(x)$ has a discontinuity at the point $x = 0$, i.e.,

$$b(x) = \begin{cases} b_0, & x > 0, \\ b_1, & x < 0, \end{cases} \qquad b_0 > b_1. \tag{1.5}$$

This discontinuity is called a bottom step. Since the mass equation (1.1) is divergent above a nonhorizontal bottom, its associated Hugoniot condition (1.3) is also satisfied at the depth discontinuity that arises above the bottom step (1.5). Since the discontinuity (1.5) is stationary ($D = 0$), Eq. (1.3) leads to the obvious condition

$$[q] = 0 \quad \Rightarrow \quad q_1 = q_0 = q(0), \tag{1.6}$$

which implies continuity of the flow rate over the bottom step. At the same time, the right side of the total momentum equation (1.2) at the discontinuity (1.5) becomes undetermined. Therefore, within the framework of formal "shallow water" theory, the Hugoniot condition (1.4) corresponding to this equation cannot be used to derive the second relation at the discontinuity (1.5). As a result, there is a need to derive the missing condition at the bottom step.

A similar situation takes place in studies of one-dimensional gas flows [5] in a tube with discontinuous cross-sectional area. Solving the discontinuity decay problem (Riemann problem)for a cross-sectional area jump,

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Dulov [6] and Yaushev [7] used, as an additional jump condition, the momentum equation in which the response of the wall p' between pipelines of different diameters was taken into account on the basis of various physical considerations (beyond the one-dimensional gas-dynamic model). However, the jump condition obtained in such a manner is determined ambiguously: it depends substantially on the method of assigning the quantity p' (in [6, 7], this quantity is specified differently). In addition, this relation is generally nondivergent, i.e., it cannot be written in the form of the Hugoniot condition $[F] = 0$.

At the same time, there is a different approach [5], in which the missing condition on the cross-sectional area jump under the flow adiabaticity assumption is obtained from the differential consequence of the basic gas-dynamic equations — the law of conservation of entropy, which retains divergent form under cross-sectional area variations. The missing condition for the isentropic gas-dynamic equations is derived in a similar manner [8], by using a divergent equation of total energy on the cross-sectional area jump along with the continuity equation.

In the present work, following the approach of [5, 8], we obtain relations at the discontinuity (1.5) from the classical conservation laws of system (1.1),(1.2), which can be written in divergent form in the case of a nonhorizontal bottom.

2. Exact Conservation Laws for the Case of a Nonhorizontal Bottom. A differential consequence of the basic equations (1.1) and (1.2) is the law of conservation of local momentum

$$v_t + (v^2/2 + z)_x = 0, \tag{2.1}$$

where $z = b + h$ is the water line. This conservation law is obtained by subtraction of Eqs. (1.1) multiplied by v from Eq. (1.2) and subsequent reduction by the positive quantity $h(x,t)$. Since Eq. (2.1) is divergent in the case of a nonhorizontal bottom, from the condition, the corresponding Hugoniot condition on the stationary discontinuity (1.5) leads to the relation

$$[v^2/2 + z] = 0, \tag{2.2}$$

which implies conservation of Bernoulli's constant above the bottom step.

As is known [3], along with (2.1), system (1.1), (1.2), like any hyperbolic system of two equations [5], admits an infinite number of linearly independent conservation laws. To obtain them, it is convenient to use system (1.1), (2.1) in vector form, which is equivalent to the system (1.1), (1.2) for smooth solutions:

$$\boldsymbol{u}_t + A(\boldsymbol{u})\boldsymbol{u}_x = \boldsymbol{f}(\boldsymbol{u}). \tag{2.3}$$

Here

$$\boldsymbol{u} = \begin{pmatrix} h \\ v \end{pmatrix}, \quad A(\boldsymbol{u}) = \begin{pmatrix} v & h \\ 1 & v \end{pmatrix}, \quad f(\boldsymbol{u}) = \begin{pmatrix} 0 \\ -b_x \end{pmatrix}. \tag{2.4}$$

Multiplying system (2.3) on the left by the gradient $U_{\boldsymbol{u}}$ of the function $U(\boldsymbol{u})$ satisfying the vector relation

$$U_{\boldsymbol{u}} A(\boldsymbol{u}) = F_{\boldsymbol{u}}, \tag{2.5}$$

where $U(\boldsymbol{u})$ and $F(\boldsymbol{u})$ are the sought scalar functions, we obtain the following conservation law:

$$U_t + F_x = -U_v b_x. \tag{2.6}$$

Eliminating the function F from system (2.5), which, with allowance for (2.4), can be written as

$$vU_h + U_v = F_h, \qquad hU_h + vU_v = F_v,$$

we obtain the second-order hyperbolic equation

$$U_{vv} = hU_{hh}, \tag{2.7}$$

which has an infinite number of linearly independent solutions $U(h,v)$, each of which corresponds to a particular conservation law of the form (2.6). Among these classical conservation laws, we distinguish those which, similarly to (2.1), can be written in divergent form

$$U_t(h,v) + \Psi_x(h,v,b) = 0 \tag{2.8}$$

and, thus, [3] are exact conservation laws, from which one can obtain the Hugoniot conditions $[\Psi(h,v,b)] = 0$ on the discontinuity (1.5).

Let us supplement system (2.3) by the obvious equation $b_t = 0$ and consider the extended system

$$\boldsymbol{w}_t + B(\boldsymbol{w})\boldsymbol{w}_x = 0, \tag{2.9}$$

where

$$\boldsymbol{w} = \begin{pmatrix} h \\ v \\ b \end{pmatrix}, \quad B(\boldsymbol{w}) = \begin{pmatrix} v & h & 0 \\ 1 & v & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In order that system (2.9), hence, system (2.3) admit the conservation law (2.8), it is necessary that the functions U and Ψ included in it satisfy the vector relation $U_{\boldsymbol{w}} B(\boldsymbol{w}) = \Psi_{\boldsymbol{w}}$, which can be written in extended form

$$vU_h + U_v = \Psi_h, \qquad hU_h + vU_v = \Psi_v, \qquad U_v = \Psi_b. \tag{2.10}$$

Eliminating the function Ψ from the first two equations of system (2.10), we arrive at relation (2.7), and eliminating this function from the other two pairs of Eqs. (2.10), we obtain

$$U_{vh} = vU_{hb} + U_{vb}, \qquad U_{vv} = hU_{hb} + vU_{vb}. \tag{2.11}$$

Because the exact conservation laws (2.8) are sought among the classical conservation laws (2.6) of system (1.1), (1.2), in which the function U does not depend on b, i.e., $U_b = 0$, it follows that from (2.11) and (2.7) we have

$$U_{vh} = U_{vv} = U_{hh} = 0 \quad \Rightarrow \quad U = C_1 h + C_2 v,$$

where $C_1 = \text{const}$ and $C_2 = \text{const}$. This implies that the set of all exact conservation laws (2.8) of system (1.1), (1.2) is a linear combination of Eqs. (1.1) and (2.1). In other words, the laws of conservation of mass (1.1) and local momentum (2.1) form a complete set of linearly independent exact conservation laws of the form (2.8) admitted by system (1.1), (1.2), and, hence, relations (1.6) and (2.2) are uniquely determined from the conservation laws (2.8).

3. Conservation Laws for the Fluid Flow over a Bottom Step. We consider a mathematical "shallow water" model in which relations (1.6) and (2.2) hold at the depth discontinuity (1.5). This means that the quantities q and $Q = v^2/2 + z$, and, hence, any continuous function of these variable are continuous at the discontinuity (1.5), i.e., at it,

$$[\eta(q, Q)] = 0 \quad \forall\ \eta(q, Q) \in C. \tag{3.1}$$

Let us elucidate which relations of the form (3.1), apart from the obvious linear combinations of conditions (1.6) and (2.2) can be obtained as integral consequences of the classical evolutionary conservation laws (2.6) of system (1.1), (1.2). In other words, we distinguish all conservation laws (2.6) that, without being exact, yet remain conservation laws for the fluid flow over a bottom step.

From (3.1) it follows that the conservation law (2.6) for the fluid flow over a bottom step is satisfied if the function included in it $U(h, v)$ meets the condition

$$U_v = \eta(q, Q), \tag{3.2}$$

where $\eta(q, Q)$ is a smooth function. Since $U_{vb} = 0$, from (3.2) we have

$$\eta_b(q, Q) = \eta_Q Q_b = \eta_Q = 0,$$

and, thus,

$$U_v(h, v) = \eta(q). \tag{3.3}$$

In this case, Eq. (2.6) can be written as an "inexact" conservation law at the discontinuity (1.5)

$$U_t + (F + \eta(q)b)_x = b\eta_x,$$

from which we have the Hugoniot condition at this discontinuity:

$$[F + \eta(q)b] = 0. \tag{3.4}$$

An important example of such a conservation law is the law of conservation of total energy

$$e_t + (q(v^2/2 + h))_x = -qb_x, \tag{3.5}$$

where

$$e = (qv + h^2)/2. \tag{3.6}$$

To derive this law, it is necessary to combine Eq. (1.2) multiplied by v and Eq. (2.1) multiplied by q. As a result, we obtain the equation of kinetic energy

$$(qv/2)_t + (qv^2/2)_x + qz_x = 0,$$

to which we add Eq. (1.1) multiplied by h and arrive at Eq. (3.5). At the discontinuity (1.5), the corresponding Hugoniot condition (3.4) has the form $[q(v^2/2 + z)] = 0$.

To derive all conservation laws (2.6) that remain conservation laws for the fluid flow over the bottom step (1.5), it is necessary to find all functions $U(h, v)$ that simultaneously satisfy the hyperbolic equation (2.7) and the additional constraint (3.3). Integrating Eq. (3.3) over v, we obtain

$$U(h, v) = \varphi(q)/h + \psi(h), \qquad q = hv, \tag{3.7}$$

where $\varphi(q) = \displaystyle\int \eta(q)\, dq$ and $\psi(h)$ is a certain function of h. From (3.7) it follows that

$$U_{vv} = h\varphi'', \qquad U_{hh} = (h^3\psi'' + q^2\varphi'' - 2q\varphi' + 2\varphi)/h^3.$$

Substituting these values of the derivatives into Eq. (2.7), we obtain

$$q^2\varphi'' - 2q\varphi' + 2\varphi = h^3(\varphi'' - \psi''). \tag{3.8}$$

Since the function φ depends only on q and ψ only on h and the velocity v is not explicitly included in Eq. (3.8), it is convenient to consider the depth h and flow rate q in this equation as independent variables. In view of this, the left side of Eq. (3.8) does not depend on h, and hence, its right side, does not depend on h either, i.e.,

$$(h^3(\varphi''(q) - \psi''(h)))'_h = 3h^2(\varphi'' - \psi'') - h^3\psi''' = 0.$$

From this it follows that

$$3\varphi'' = h\psi''' + 3\psi''.$$

The left side of this equation depends only on q, and its right side only on h. Therefore,

$$3\varphi'' = C, \qquad h\psi''' + 3\psi'' = C, \qquad C = \text{const}.$$

Integrating these equations, we obtain

$$\varphi = C_1 q^2 + C_2 q + C_3, \qquad \psi = C_1 h^2 + C_4 h + C_5/h + C_6, \tag{3.9}$$

where $C_1 = C/6$ and $C_i = \text{const}$. Since function (3.7) is defined by Eqs. (2.7) and (3.3) with accuracy to an arbitrary constant term, the constant C_6 included in (3.9) can be dropped.

From (3.9) it follows that

$$\varphi' = 2C_1 q + C_2, \qquad \varphi'' = 2C_1, \qquad \psi'' = 2(C_1 + C_5/h^3).$$

Substituting these values of the derivatives into the basic equation (3.8) and collecting terms, we obtain

$$C_3 + C_5 = 0. \tag{3.10}$$

Substituting functions (3.9) into (3.7) and taking into account (3.10), we find the general form of the function $U(h, v)$ that satisfy Eqs. (2.7) and (3.3):

$$U = C_1(qv + h^2) + C_2 v + C_4 h.$$

From this it follows that these equations admit only three linearly independent solutions:

$$U_1 = h, \qquad U_2 = v, \qquad U_3 = e = (qv + h^2)/2,$$

which correspond to the laws of conservation of mass (1.1), local momentum (2.1) and total energy (3.5). Thus, Eqs. (1.1), (2.1), and (3.5) form a complete system of linearly independent classical conservation laws (2.6) for the system of “shallow water” equations (1.1) and (1.2) satisfied at the depth discontinuity (1.5).

4. Derivation of an Additional Relation on a Bottom Step Using a Limiting Transition from a Higher-Level Model. As shown in Sec. 3, from the Hugoniot conditions admitted by the "shallow water" equations at the depth discontinuity it follows that the mass and total energy of the flow are conserved at the bottom step. If the conservation of mass at the discontinuity (1.5) is unquestionable from a physical viewpoint, the conservation of total energy (3.6) seems to be insufficiently justified. Indeed, if we assume that function (1.5) describes a discontinuity of a real bottom surface, a vertical step (at $[b]v < 0$), or a vertical weir (at $[b]v > 0$)) forms in the flow over the discontinuity. In both cases (for head-on impact of water on the vertical wall of the step or for its impact on the bottom as a result of flow down a vertical weir) a considerable portion of the total energy of the flow can be converted to the energy of internal vortex and turbulent mixing, which implies loss of energy from the viewpoint of "shallow water" theory. Thus, the "total" energy (3.6), which is the sum of only the kinetic and potential energies of the flow may not be conserved for water flows over real vertical bottom steps. Therefore, the above closure of the "shallow water" model at the discontinuity (1.5) is generally not intended for description of such flows. This is due to the fact that classical "shallow water" theory is formulated using the long-wave approach of the hydrodynamic equations [1–4], in which all horizontal dimensions of flow should be much larger than its vertical dimensions (in particular, the wavelengths should be much greater than the average flow depth). By virtue of this, the classical discontinuous solutions of the Saint Venant equations (1.1) and (1.2) — hydraulic bores — are actually transition regions (of continuous but rather rapid variation of flow parameters) whose width far exceeds the characteristic flow depths.

Similarly, from a physical viewpoint, the discontinuity (1.5) of the depth $b(x)$ in the "shallow water" model is a transition zone $[-\varepsilon, \varepsilon]$ of rather rapid but smooth and monotonic variation of real bottom depth:

$$b(\varepsilon, x) = \begin{cases} b_0, & x \geqslant \varepsilon, \\ \bar{b}(x), & -\varepsilon \leqslant x \leqslant \varepsilon, \\ b_1, & x \leqslant -\varepsilon, \end{cases} \qquad b(\varepsilon, x) \in C^1, \tag{4.1}$$

whose width far exceeds the maximum flow depth, i.e., $2\varepsilon \gg \max h(x)$. Then, the assumption of conservation of total flow energy on the segment $[-\varepsilon, \varepsilon]$ of such a real channel is quite natural. This means that the "shallow water" model with the Hugoniot condition (1.6) and (2.2) at the bottom step (1.5) describes transients on rather smooth steps and weirs in a real channel. From this, in turn, it follows that the "shallow water" equations (1.1) and (1.2) with a "spread" discontinuity of depth (4.1) is a model of higher level than the same equations with discontinuous depth (1.5). Therefore, the additional relation for the bottom step (1.5) can be obtained from this "higher-order" model by passage to the limit $\varepsilon \to 0$.

Let us consider any differential consequence of the basic system (1.1), (1.2) that is linearly independent of Eq. (1.1). For a smooth steady flow over the spread bottom step (4.1), all such differential corollaries, in particular, the equations of local momentum (2.1) and total energy (3.5) are equivalent, taking into account that $q = \text{const}$. Therefore, passage to the limit $\varepsilon \to 0$ from any of them gives the same relation on the discontinuity (1.5), which coincides with the Hugoniot condition (2.2).

For Eqs. (2.1) and (3.5), this statement is obvious. Therefore, we consider the equation of total momentum (1.2), which is not a conservation law above the bottom step (1.5). For the steady-state solution, for which $q = \text{const}$ by virtue of (1.1), Eq. (1.2) can be written as

$$q^2 (h^{-1})_x + h z_x = 0.$$

After division by h, this equation can be written in divergent form

$$(v^2/2 + z)_x = 0. \tag{4.2}$$

Integrating Eq. (4.2) over the spread bottom step (4.1) from ε to $-\varepsilon$ subject to the boundary conditions $h(\varepsilon) = h_0$, $h(-\varepsilon) = h_1$ and passing then to the limit at $\varepsilon \to 0$, we obtain the Hugoniot condition (2.2).

The method of deriving conditions on the discontinuity (1.5) using steady-state solutions of the basic system with a spread singularity of its right side is apparently the most general one for problems of this type. In this case, in contrast to [6, 7], the response of the bottom step is uniquely expressed from the equation of momentum (1.2) by the formula

$$F = \int\limits_{\varepsilon}^{-\varepsilon} h b_x \, dx = \int\limits_{b_0}^{b_1} h \, db = -\Big[qv + \frac{h^2}{2} \Big] = -q^2 \Big[\frac{1}{h} \Big] - \frac{1}{2} [h^2] = [h](v_0 v_1 - \bar{h}),$$

where $\bar{h} = (h_0 + h_1)/2$.

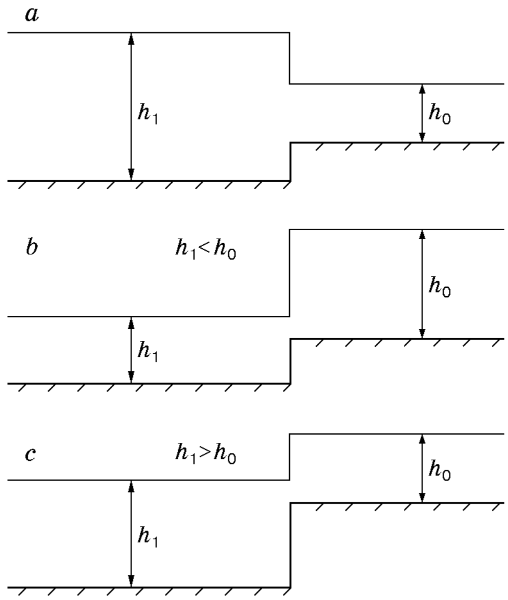

Fig. 1

5. Admissible flows over the bottom step. Because the Hugoniot condition (2.2) subject to the continuity condition of the flow rate (1.6) can be written as

$$q^2 = h_0^2 h_1^2 [z]/(\bar{h}[h]), \tag{5.1}$$

where $q = q(0) \neq 0$, it follows that only those flows over the discontinuity (1.5) are possible for which

$$[z][h] = (z_1 - z_0)(h_1 - h_0) = (h_1 - h_0 - \delta)(h_1 - h_0) > 0 \tag{5.2}$$

($\delta = -[b] = b_0 - b_1 > 0$). Inequality (5.2) distinguishes two admissible flow configurations over the bottom step: for the first of them (Fig. 1a),

$$z_1 > z_0 \sim h_1 > h_0 + \delta, \tag{5.3}$$

and for the second (Fig. 1b),

$$h_1 < h_0. \tag{5.4}$$

For comparison, Fig. 1c gives the configuration for $h_0 < h_1 < h_0 + \delta$ that is not admitted by inequality (5.2). The inadmissibility of this configuration within the framework of the model considered implies that the discontinuous solutions shown in Fig. 1c are possible only in the case of loss of total energy at the bottom step (such flows are not studied in the present paper).

From equality (5.1) it follows that the flow velocity v_1 on the left of the step (1.5), specified by the formula $v_1^2 = h_0^2[z]/(\bar{h}[h])$, satisfies the inequality $v_1^2 < h_1$ subject to condition of (5.3), and it satisfies the inequality $v_1^2 > h_1$ subject to condition (5.4). This implies that for the configuration shown in Fig. 1a, the flow on the left of the discontinuity is always subcritical ($|v_1| < c_1$; $c = \sqrt{h}$ is the velocity of sound) and for the configuration shown in Fig. 1b, it is always supercritical ($|v_1| > c_1$). The type of flow on the right of the bottom step cannot be determined directly from Eqs. (5.1). To determine this flow type, it is necessary to study the stability of the discontinuous flows (5.3) and (5.4) over the bottom step (1.5).

It should be noted that in studying the stability of the flows at the discontinuities (5.3) and (5.4), one cannot use the entropy criterion [9], which for the conservation laws for "shallow water" is related to the loss of total energy on hydraulic bore[3, 10] because in the model considered, the total energy at the bottom step is conserved. Nor can one apply the classical characteristic criterion of [9], according to which three characteristics arrive at a discontinuity line and only one characteristics leaves it. For a steady hydraulic bore, this allows one, using two Hugoniot conditions (1.3) and (1.4) and three relations for the incoming characteristics, to uniquely determine the wave propagation velocity D and the depths and velocities ahead of (h_0, v_0) and behind (h_1, v_1) the wave front. However, for a stationary discontinuity over the bottom step, for which it is necessary to evaluate only four flow parameters h_0, v_0, h_1, and v_1, such an overdetermined system of five equations generally has no solution.

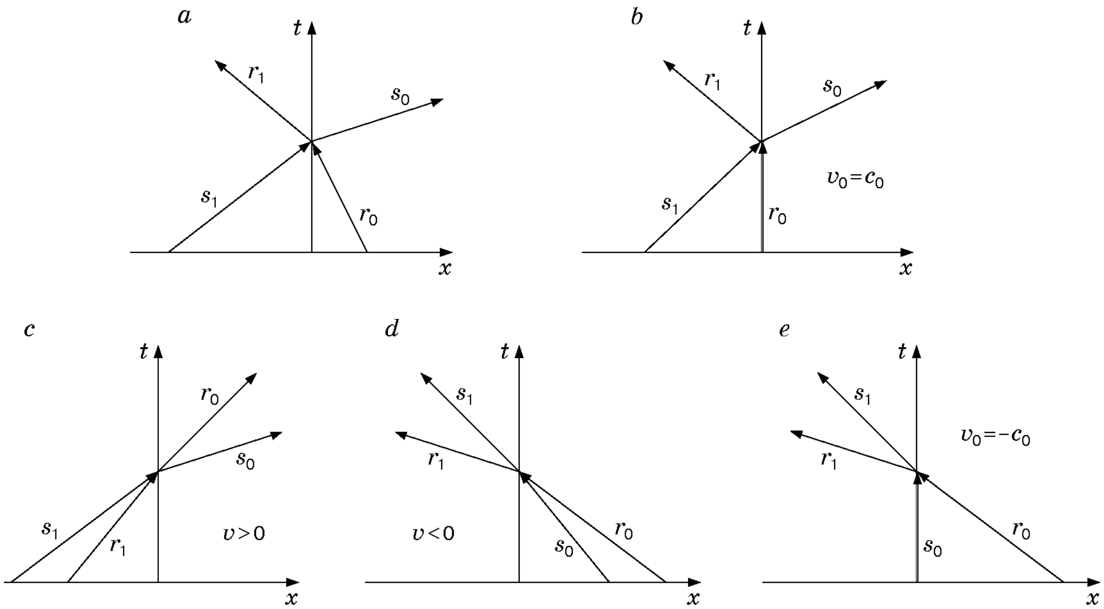

Fig. 2

To uniquely determine the parameters h_0, v_0, h_1, and v_1 on a stationary discontinuity line, it is necessary that two characteristics arrive at the discontinuity line and two characteristics be strictly outgoing (in this case, the characteristics propagating with zero velocity along the stationary discontinuity line are included in the number of incoming characteristics and are not included in the number of strictly outgoing characteristics). Exactly this evolutionarity condition [11] should be used as the stability criterion for the discontinuity above the bottom step. In the case of the configuration shown in Fig. 1a, for which the flow on the left of the front is subcritical ($|v_1| < c_1$), this means that in order that the discontinuity be stable, the flow on the right of its front should be subcritical ($|v_0| < c_0$) or critical with positive velocity ($v_0 = c_0$). In the case of the configuration shown in Fig. 1b, for which the flow on the left of the front is supercritical ($|v_1| > c_1$), in order that the discontinuity be stable, it is necessary that the flow on the right of its front be supercritical ($|v_0| > c_0$) or critical with negative velocity ($v_0 = -c_0$). Furthermore, in all cases, the flow on both sides of the discontinuity should be unidirectional ($v_0 v_1 > 0$).

Figure 2a and b shows the characteristic fields corresponding to steady flows of the first configuration (see Fig. 1a). Figure 2c–e shows the characteristic fields corresponding to steady flows of the second configuration (see Fig. 1b). Figure 2a corresponds to subcritical flow, Fig. 2b and e to critical flow, and Fig. 2c and d to supercritical flow on the right of the bottom step (1.5). The letters r and s in Fig. 2 denote the r and s invariants ($r = v - 2c$ and $s = v + 2c$) transferred, respectively, along the r and s characteristics, propagating with velocities $\lambda_r = v - c$ and $\lambda_s = v + c$.

Assuming that $q = \text{const} \neq 0$, we write Eq. (2.2) in the form

$$\delta = J(h_1) - J(h_0). \tag{5.5}$$

The plot of the function $J(h) = q^2/(2h^2) + h$ is given in Fig. 3. Because (see [4])

$$J'(h) = 1 - q^2/h^3 = 1 - v^2/h,$$

the function $J(h)$ reaches the minimum

$$\min J(h) = J(h_c) = 3h_c/2 = 3q^{2/3}/2$$

on the critical flow at the point $h_c = v_c^2 = q^{2/3}$, it follows that for $h < h_c$, for which $J'(h) < 0$, it corresponds to supercritical flows $|v| > c$, and for $h > h_c$, for which $J'(h) > 0$, this function corresponds to subcritical flows $|v| < c$. This implies that for steady flows with given flow rate q, Eq. (5.5) is uniquely solvable for h_1 for $h_0 > 0$ and is relatively solvable for h_0 for all values of h_1 that satisfy the inequality

$$J(h_1) = v_1^2/2 + h_1 > \delta.$$

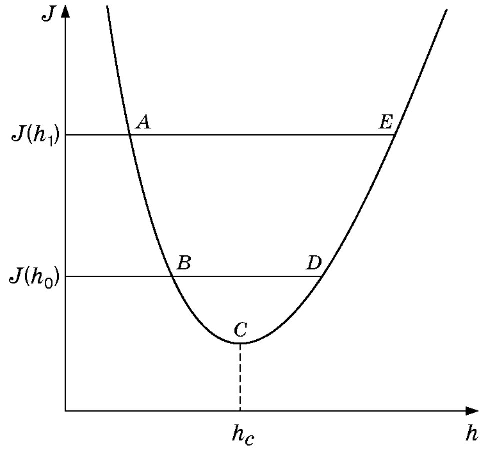

Fig. 3

In this case, sable discontinuities whose fields of characteristics are shown in Fig. 2 correspond to the jumps on the plot of the function $J(h)$ (see Fig. 3): the jump DE corresponds to Fig. 2a, the jump CE at $q > 0$ to Fig. 2b, the jump BA to Fig. 2c and d, and the jump CA at $q < 0$ to Fig. 2e.

Relation (5.5) can be written as a cubic equation for both h_1, and h_0. Therefore, using Cardan's formula, it is possible to obtain explicit dependences $h_1(h_0, q)$ and $h_0(h_1, q)$ for the stable discontinuities considered. For example, for the jump DE, located in the subcritical flow domain (see Fig. 3), the indicated dependence has the form

$$h_1 = a\Big(1 + 2\cos\Big(\frac{1}{3}\arccos\frac{a^3 - q^2/4}{a^3}\Big)\Big), \tag{5.6}$$

where $a = (J(h_0) + \delta)/3 = (q^2/(2h_0^2) + h_0 + \delta)/3 = (v_0^2 + 2z_0)/6$.

6. Problem of the Flow Arising from Dam Failure above a Step. We use the above results to solve the problem of the flow arising from dam failure above the bottom step (1.5) over which water is flowing. That is, we consider the problem of decay of the initial discontinuity of levels

$$z(x,0) = \begin{cases} z_0, & x > 0, \\ z_1, & x < 0, \end{cases} \qquad z_1 > z_0 \tag{6.1}$$

above the bottom step (1.5) in originally quiescent water:

$$v(x,0) = 0. \tag{6.2}$$

This problem is a special case of the general problem of arbitrary discontinuity decay (Riemann problem) above the bottom step (1.5), whose analog for the equations of gas dynamics was considered in [5–8]. The classical Riemann problem for the "shallow water" equations above a horizontal bottom was studied in [12, 13].

As is known, the solution of problem (6.1), (6.2), (1.5) can be sought as a combination of simple waves (i.e., shocks propagating with constant speed and rarefaction waves centered about the coordinate origin), the stationary jump located at the coordinate origin above the bottom step, and constant flow regions in between them. In this case, from the initial conditions (6.1) and (6.2) it follows that an S shock propagates to the right of the initial background z_0 and a rarefaction R wave propagates to the left of the initial background z_1 (we recall that an S shock is a shock wave at which two s characteristics arrive, and a rarefaction R wave is a centered rarefaction wave in which the r invariant changes and the s invariant is constant).

After decay of the discontinuity (6.1), (6.2), the fluid flows in the positive direction ($v > 0$). Therefore, on the left of the discontinuity above the bottom step (this stationary discontinuity will be denoted by L), a region of constant subcritical flow forms, which connects the discontinuity L with the rarefaction R wave. This implies that in the solution of problem (6.1), (6.2), the configuration (5.3) shown in Fig. 1a [6, 7] always occurs at the discontinuity L. Therefore, the flow that arises above the bottom step can be called sill flow, according to the terminology of [6, 7].

The flow pattern on the right of the discontinuity L depends significantly on the type of flow behind the S shock front. Let us elucidate the conditions under which this flow is subcritical. Taking into account that the fluid

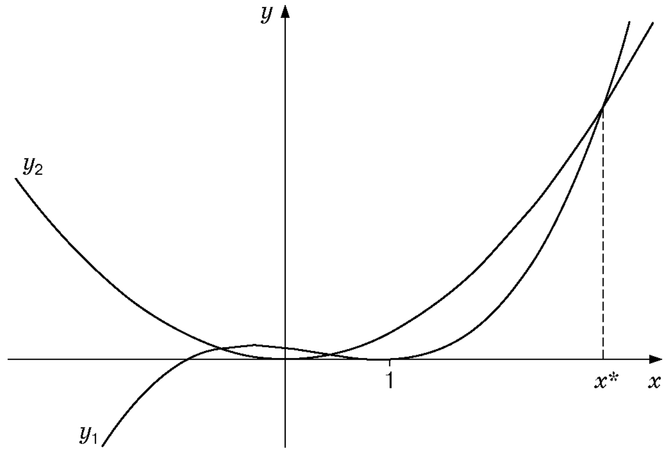

Fig. 4

ahead of the S shock is at rest ($v_0 = q_0 = 0$), we can write the Hugoniot conditions (1.3) and (1.4) ahead of its front in extended form

$$q_2 = D(h_2 - h_0), \qquad Dq_2 = q_2 v_2 + (h_2^2 - h_0^2)/2, \tag{6.3}$$

where the subscript 2 corresponds to flow parameters behind the shock. Using Eqs. (6.3) for specified depths h_0 and h_2, we uniquely calculate the velocity of the S shock $D > 0$

$$D^2 = h_2(h_2 + h_0)/(2h_0)$$

and the flow rate $q_2 > 0$ behind the wave front

$$q_2^2 = h_2(h_2 + h_0)(h_2 - h_0)^2/(2h_0). \tag{6.4}$$

From (6.4) it follows that the flow behind the S shock front is subcritical provided that

$$v_2^2 = q_2^2/h_2^2 = (h_2 + h_0)(h_2 - h_0)^2/(2h_0h_2) < c_2^2 = h_2.$$

Converting this condition, we obtain

$$y_1 = (x + 1)(x - 1)^2 < y_2 = 2x^2, \tag{6.5}$$

where $x = h_2/h_0 > 1$.

From the plots of the cubic $y_1(x)$ and quadratic $y_2(x)$ functions, presented in Fig. 4, it follows that at $x > 1$, i.e., for the steady shock, the solution of the cubic inequality (6.5) has the form $1 < x < x^*$, where

$$x^* = 1 + \frac{4}{\sqrt{3}} \cos\Big(\frac{1}{3} \arccos \frac{3\sqrt{3}}{8}\Big) \approx 3.214$$

is the maximum root of the cubic equation $x^3 - 3x^2 - x + 1 = 0$, calculated by Cardan's formula. This means that the flow behind the front of the S shock is subcritical at $h_0 < h_2 < h_0^*$ ($h_0^* = x^* h_0$), critical at $h_2 = h_0^*$, and supercritical at $h_2 > h_0^*$.

As shown in Sec. 5, for steady flows with specified flow rate q, the Hugoniot condition (5.1) on the discontinuity L is uniquely solvable for h_1. Therefore, further solution of problem (6.1), (6.2), (1.5) is constructed from its right background value of the depth $h_0 = z_0 - b_0$ to the left background value $h_1 = z_1 - b_1$, which will be considered a priori unknown (in this case, along with h_0, the depth $h_2 > h_0$ behind the front of S shock is also considered known).

We first assume that the flow behind the S shock front is subcritical or critical, i.e., $h_0 < h_2 \leqslant h_0^*$. Then, the constant flow (h_2, v_2) continues immediately to the discontinuity L above the bottom step (Fig. 5a). As a result, a steady flow forms on it, whose fields of characteristics are shown in Fig. 2a and b. The value of the depth h_3 behind the discontinuity L is uniquely defined by the formula (5.6), where $h_1 = h_3$, $h_0 = h_2$, and $q = q_2$. The initial depth h_1 behind the bottom step is then calculated by the formula

$$h_1 = (\sqrt{h_3} + v_3/2)^2 = (\sqrt{h_3} + q_2/(2h_3))^2,$$

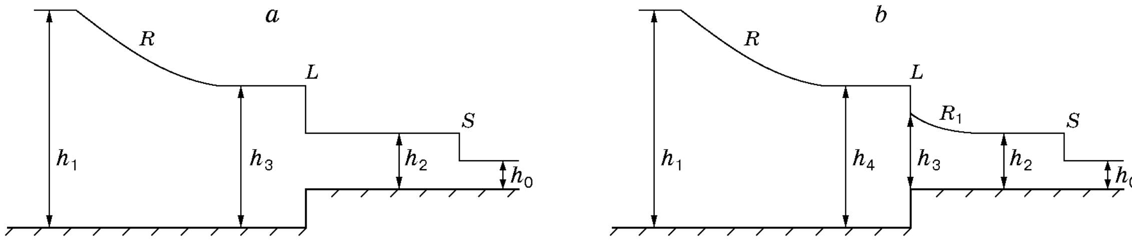

Fig. 5

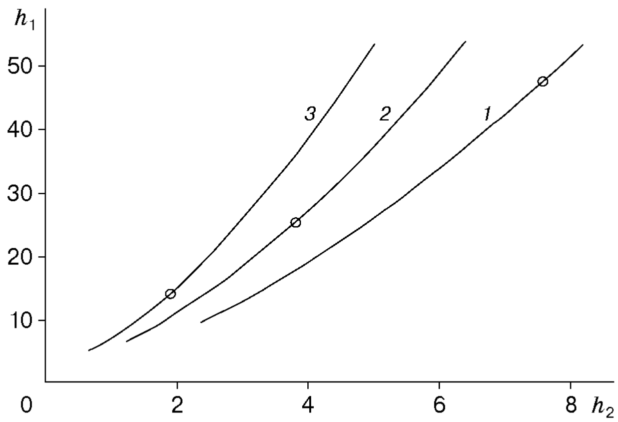

Fig. 6

obtained from the condition of constancy of the s invariant in the rarefaction R wave:

$$s = 2\sqrt{h_1} = v_3 + 2\sqrt{h_3}, \qquad v_3 = q_2/h_3.$$

Let us now assume that the flow behind the front of the discontinuous S wave is supercritical, i.e., $h_2 > h_0^*$. Because under the condition (5.3) (see Fig. 1a), the supercritical flow ahead of the discontinuity L is unsteady, the flow (h_2, v_2) cannot be continued immediately to the bottom step. By virtue of this, it is converted to an additional rarefaction R_1 wave, whose left boundary adjoins the discontinuity L. This R_1 wave, together with the discontinuity L, forms a unified LR_1 wave jump (Fig. 5b). The flow (h_3, v_3) on the left boundary of the rarefaction R_1 wave is critical:

$$v_3 = c_3 = \sqrt{h_3}. \tag{6.6}$$

This leads to formation of a steady flow at the discontinuity L. The field of characteristics for this flow is shown in Fig. 2b. In this case, the velocity v_3 and depth h_3 are calculated by the formulas

$$v_3 = (v_2 + \sqrt{h_2})/3, \qquad h_3 = v_3^2,$$

which are obtained with allowance for (6.6)from the condition of constancy of the s invariant in the rarefaction R_1 wave

$$s = v_3 + 2\sqrt{h_3} = 3v_3 = v_2 + 2\sqrt{h_2}.$$

Then, the flow (h_4, v_4) behind the discontinuity L and the required initial depth h_1 are recovered similarly to the previous case.

Thus, it is shown that the solution of problem (6.1), (6.2) above the bottom step (1.5), for which, without loss of generality, we can assume that $\delta = -[b] = 1$, is uniquely recovered for any values of the depths $h_2 > h_0 > 0$ at the S shock. This implies the existence of a function

$$h_1 = h_1(h_2, h_0) \tag{6.7}$$

determined for all $h_2 > h_0 > 0$. From this it follows that to prove the unique solubility of the problem of dam failure (6.1), (6.2), (1.5), it will suffice to show that for all $h_0 > 0$, the function (6.7) increases strictly monotonically with respect to h_2 for all $h_2 > h_0$ and satisfies the conditions

$$\lim_{h_2 \to h_0} h_1(h_2, h_0) = \delta + h_0 = 1 + h_0, \qquad \lim_{h_2 \to \infty} h_1(h_2, h_0) = +\infty. \tag{6.8}$$

In this case, at $h_0 > 0$ and $h_1 > 1 + h_0$ there exists an inverse function $h_2 = h_2(h_1, h_0)$, which is used to calculate the depth h_2 behind the front of the S shock, and then to uniquely recover the remaining parameters of the solution of the problem of dam failure above the step.

Formal proof of the rigorous monotony of the function (6.7) from h_2 is cumbersome. The monotony of this function is supported by computed curves of $h_1(h_2, h_0)$ for $h_0 = 0.5$, 1, and 2 (curves 1–3 in Fig. 6). The points on these curves show the values of h_0^*, for which the flow behind the front of the S shock is critical. From Fig. 6 it is evident that for the indicated values of h_0 the function (6.7) increases strictly monotonically and obeys (6.8).

In conclusion, we are planning to consider the solution of the general Riemann problem above a bottom step using the results obtained in the present work and the methods developed in [6, 7, 12, 13].

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